Application of CAS to the classical definition of limit of function

(*An alternative approach to introduce the concept of limit*)

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Applications of Computer Algebra

E'TS, Montreal, CANADA

Classical Definition

$$\lim_{x \to a} f(x) = L$$

$$\Downarrow$$

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that}$$

$$|x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Tiny Creature (ε, δ) → **Great Fear at the start of Course**

Standard Question

For a given function *f*, a point *a* and $\varepsilon > 0$

verify by using the ε-δ **difination that**

$$\lim_{x \to a} f(x) = L$$

Examples

1. When f is a constant funtion:

   Ans. Any real number $\delta > 0$ satisfies the definition.

2. When f is a linear funtion, e.g. $f(x) = 5x - 4$

   Ans. Any real number $0 < \delta \leq \frac{\varepsilon}{5}$ satisfies the definition.

3. What about when f is a non-linear funtion,

e.g. $f(x) = x^2$ L = 9 $a = 3$.

(A simplest non-linear function)

Solution as given in most of the Calculus Textbooks

i. Using $(\Delta, \le)$: $|x+3| = |(x-3)+6| \le |x-3| + 6$

ii. $0 < |x-3| < \delta \Rightarrow |x+3||x-3| \le |(x-3)+6||x-3| < (\delta+6)\delta$

iii. Need $\delta > 0$ so that $(\delta+6)\delta \le \varepsilon$

iv. Restrict our attention to δ so that $\delta \le 1$

v. Apply (iv) to δ appearing in the 1st factor: $(\delta+6)\delta \le 7\delta$

vi. Note: "$0 < |x-3| < \delta \Rightarrow |x+3||x+3| < \varepsilon$" holds as long as $(\delta+6)\ \delta < 7\delta$

vi. *Answer* : Choose $0 < \delta = \min(\frac{\varepsilon}{7}, 1)$

Can a student handle Tricky Inequalities at the Freshman level?

What about if $f(x)$ is a Rational Function**?**

Many students start hating Calculus at this early stage!

Main Point of Difficulty

Finding $\delta > 0$ to arrive at a **symmetric interval**

$$(a-\delta, a+\delta)$$

about the Point a

A Remedy by a Simple APPROXIMATION PROBLEM

Modification in $\varepsilon - \delta$ definition of limit (2006, IJMEST)

Given a function f, a point a, $\varepsilon > 0$ and a number L find the largest interval $I_{\varepsilon,a}$ *such that*

i. $I_{\varepsilon,a}$ **contains** a

ii. **Deviation of f from is at most L**

i.e.

$$x \in I_a \setminus \{a\} \Rightarrow |f(x) - L| < \varepsilon$$

Aapproach (Use of CAS)

Find **two real zeros** (if any) **of the fucntions**

$$\left.\begin{aligned} g_1(x) &= f(x) - L + \varepsilon \\ g_2(x) &= f(x) - L - \varepsilon \end{aligned}\right\}$$

such that **one is to the left** and **other to the right** of a.

Concept

We shall call L an ε - Local Approximation of f at a

Alternative Definitin of Limit

$$\lim_{x \to a} f(x) = L$$

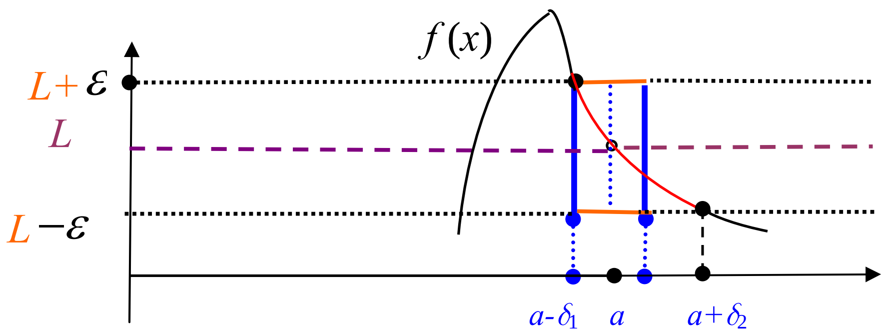
if and only if

L is an ε - Local Approximation of f at a *for all* $\varepsilon > 0$

Graphical Interprtation in case of

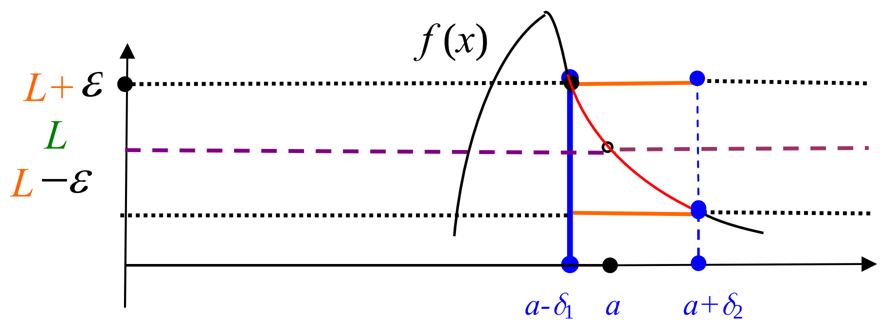
(i) Classical Definition of Limit

(Here $0 < \delta \leq \min(\delta_1, \delta_2)$)



(ii) Alterntive Definition of Limit

(Here $I_{\varepsilon,a} = (a - \delta_1, a + \delta_2)$)



Contra-positive statement

A number L is not the limit of $f(x)$ as $x \to a$ if there exists a strip bounded by $y = L \pm \varepsilon$ in the xy-plane such that a part of the graph of $f(x)$ over any punctured interval $I_a \setminus \{a\}$ lies outside the strip.

Typical Question:

Find the largest interval $I_{\varepsilon,a}$ which assures that L is an ε - Local Approximation of f at a.

Answer

The end points c and d of $I_{\varepsilon,a} = (c,d)$ can be determined by the following steps:

Step 1. Solve each of the following equations separately (**use of CAS**):

$$\left.\begin{array}{l} f(x) - L + \varepsilon = 0 \\ f(x) - L - \varepsilon = 0 \end{array}\right\} \qquad (1)$$

Suppose that the resultant **real zeros in both equations** of (1) are

$$z_1, z_2, z_3, \ldots \qquad (2)$$

Step 2. Find two zeros $c, d \in \{z_1, z_2 \ldots\}$ with the properties (**use of CAS**):

- c is to the left and d is to the right of a
- ***c* and *d*** are the two **closest points** to ***a*** as compared to the other real zeros, i.e.,

$$\left.\begin{array}{l} |c - a| \le \min\{z_i : i = 1,2,\ldots; z_i \text{ is to the Left side of } a\}, \\ |d - a| \le \min\{z_i : i = 1,2,\ldots; z_i \text{ is to the Right side of } a\} \end{array}\right\} \quad (3)$$

Step 3. The required interval, the largest one satisfying the Property A is given by

$$I_{\varepsilon,a} = (c,d) \qquad (4)$$

The use of CAS is imminent in finding the answer when the Function is non-trivial.

Contra-positive statement

A number L is not the limit of $f(x)$ as $x \to a$ if there exists a strip bounded by $y = L \pm \varepsilon$ such that a part of the graph of function over any puncture interval $I_a \setminus \{a\}$ does not lie within the strip

Some Comments about the Modified Definition

1. **Comments after 1st Presentation at KFUPM**

   i. ***Dr. Bokhari! The conventional definition has been considered over the decades. Do you think that it can be replaced by your thoughts?***

   ii. ***What you have presented is available in the common calculus textbooks. It is not something new.***

2. **Reviewer's Comments (IJMEST, 2006):**

   I like the paper and I think many other calculus instructors will take this alternative approach to rigorously proving that limit exists.

Classroom Experiment and Pedagogical Implications

Experiment (at KFUPM, Dhahran, Saudi Arabia):

31 Freshman students (Calculus I during the semester 082)

Number of Lectures:

2 (1st for Classroom Lecture, 2nd for CAS, Quiz & Survey)

Nature of Lecture:

Use of Overhead projector & MAPLE (for finding roots)

Period of Experiment:

11th week of Classes

Students had already gone through the classical definition of limit during the first week of classes

Class Instructor:

Dr. Hussain Al-Attas who allowed us to perform the experiment.

Class Size:

31 students

Methodology:

Our **presentation** and **survey** was confined to two recitation classes (50 minutes each). In the **first class,** we intoduced the alternative definition of limit after a brief description of the classical definition in case of one-variable functions which they had already received in their earlier lessons. Both definitions, classical and alternative, were explained with the help of some nontrivial examples.

In the **second class**, the students were introduced on how to use software to find the roots of the equations (2) required in the modified definition of limit. The students were also tested on how to find the limit using the new definition. This experiment was wound up by conducting a survey that aimed at determining the students' viewpoint about the modified definition in comparison to the classical one.

Students' Response:

The survey revealed that almost **40%** of the students **could not understand the classical definition of limit.** However, **71%** of the students reported that **they understood the alternative definition of limit.** **65%** of the students indicated that the **new definition is easier than the classical definition**.

Note

(i) Some of the students among 29% who could not follow the alternative definition did not attend the first class in which the definition was explained.

(ii) It may be taken into consideration that students had seen the alternative definition first time in the class and that it is not a part of the textbook.

(iii) Most of the students appreciated this definition at the end of experiment.

(iv) The results indicates that the alternative definition has a good potential to replace the classical definition, and that the students did not find it difficult as compared to the classical definition.

Use of CAS (Studens' Response)

Regarding finding the zeros of the two equations (cf (2)), students showed a lot of interest when its demonstration was given through MAPLE MATLAB and MATHCAD. In the survey, almost 78% of the students indicated that **they can easily find the zeros of the equations using MAPLE**.

After finding the zeros, more than **62%** of the students' indicated that **they can use the zeros to find the delta in the classical definition.**

Another aspect of the survey was related to the 'importance of the limit' as deemed by the students. To this, more than **65%** of the students indicated that the **alternative definition with the use of CAS helped them to understand the importance of the limit.**

On-going Work

We are extending the alternative definition of limit for the functions of 2 and 3 variables where we shall demonstrate an intensive use of CAS.

Alternative definition of limit for a function of two variables

Concept: For a given $\varepsilon > 0$, a number L is an ε-Local Approximation of f at (x_0, y_0) if f satifies the following property:

"*There exists an annulus region or strip* $R_{\varepsilon,(x_0,y_0)}$ (with linear or curved boundaries) containing (x_0, y_0) in the *x*y-plane *such that* deviation of f from L is at most ε *over* $R_{\varepsilon,(x_0,y_0)} - \{(x_0, y_0)\}$.

Definition:

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

if and only if

L is an ε - Local Approximation of f at (x_0, y_0) *for all* $\varepsilon > 0$.

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Reference

[1] M. A. Bokhari and B. Yushau, "Local (L, ε)-approximation of a function of a single variable: an alternative way to define limit," Intern. J. Math. Educ. in Sciences and Technology, Vol. 37 (2006) pp. 515-526.