Uniqueness of certain polynomials constant on a hyperplane

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Motivation

I will begin with the motivation from complex analysis. Let

$$z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$$

be the coordinates or ($(z, w) \in \mathbb{C}^2$). Let

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An interesting question asked a long time ago (by Rudin I believe) is:

Question

What are the proper holomorphic maps from $\mathbb{B}_n$ to $\mathbb{B}_N$ ($n \neq N$ in general)

Proper means that f^{-1} takes compacts to compacts. If f extends to the boundary, proper f takes the boundary to the boundary.
The answer is not completely known even if you restrict the maps to be rational, polynomial, or even monomial.

A sample of what is known

Let $f\colon \mathbb{B}_n \to \mathbb{B}_N$ be a proper holomorphic map.

Exercise

If $n = N = 1$ (I.e. maps taking circle to circle), then f is a finite Blaschke product. That is, $z \mapsto e^{i\theta} \prod_{j=1}^{k} \frac{z-a_j}{1-\bar{a}_j z}$.

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If $n > N$, no proper map exists.

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From now on $n \geqslant 2$.

Theorem (Alexander '77 (also Pelles, Pinchuk, Fornaess))

If $n = N$, $n \geqslant 2$, then f is an automorphism of $\mathbb{B}_n$.

A sample of what is known II

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Theorem (Rudin '84 (much easier proof by D'Angelo))

If f is a homogeneous polynomial then f is equivalent to $z \mapsto z^{\otimes d}$.

Where $z^{\otimes d}$ is the symmetric tensor with proper weights. E.g. for $n = 2$, $d = 3$ we have

$$(z, w) \mapsto (z, w)^{\otimes 3} = (z^3, \sqrt{3}z^2 w, \sqrt{3}zw^2, w^3).$$

The equivalence is up to automorphisms of $\mathbb{B}_n$ and $\mathbb{B}_N$.

And we get to monomial maps

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We concentrate on $n = 2$. The following is really the kind of statement we would be happiest with.

Theorem (Faran '82)

If $f \colon \mathbb{B}_2 \to \mathbb{B}_3$ is C^3 up to the boundary then f is equivalent to

1. $(z, w) \mapsto (z, w, 0)$
2. $(z, w) \mapsto (z, zw, w^2)$
3. $(z, w) \mapsto (z^2, \sqrt{2}zw, w^2)$ *(that is $(z, w)^{\otimes 2}$)*
4. $(z, w) \mapsto (z^3, \sqrt{3}zw, w^3)$

Real geometric setup

Let $f\colon \mathbb{B}_2 \to \mathbb{B}_N$ be a proper map extending to the boundary. Then

$$\|f(z,w)\|^2 = |f_1(z,w)|^2 + \cdots + |f_N(z,w)|^2 = 1 \quad \text{if} \quad |z|^2 + |w|^2 = 1$$

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Let f be monomial, that is, $f_k = c_k z^{a_k} w^{b_k}$. Then

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$$|f_k(z,w)|^2 = |c_k|^2 \left(|z|^2\right)^{a_k} \left(|w|^2\right)^{b_k}.$$

Replace $x = |z|^2$ and $y = |w|^2$. Then $\|f(z,w)\|^2$ becomes a real polynomial $p(x,y)$ with nonnegative coefficients such that

$$p(x,y) = 1 \quad \text{if} \quad x + y = 1.$$

If all monomials in f were distinct, then N is the number of monomials in $p(x,y)$.

Degree estimate

Denote by $N(p)$ the number of distinct monomials in p.

Theorem (D'Angelo, Kos, Riehl '03)

Let $p(x, y)$ be a real polynomial of degree d with nonnegative coefficients. Suppose that $p(x, y) = 1$ on $x + y = 1$. Then

$$d \leqslant 2N(p) - 3.$$

This is sharp, for odd d

$$f_d(x, y) := \left(\frac{x + \sqrt{x^2 + 4y}}{2}\right)^d + \left(\frac{x - \sqrt{x^2 + 4y}}{2}\right)^d + (-1)^{d+1} y^d$$

are polynomials with $d = 2N(f_d) - 3$.

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$$\begin{bmatrix} \epsilon & 0 \\ 0 & \epsilon^2 \end{bmatrix}$$

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  for ϵ being the d^{th} root of unity.
- My coauthor (somewhere in the room) together with D'Angelo proved that these groups together with the group generated by ϵI are essentially the only group representations allowing an invariant proper map of balls.

Sharp polynomials

Let $\mathcal{H}(2, d)$ be the set of degree d polynomials $p(x, y)$ with nonnegative coefficients such that $p(x, y) = 1$ when $x + y = 1$.

Call $p \in \mathcal{H}(2, d)$ *sharp* if p minimizes $N(p)$ in $\mathcal{H}(2, d)$.

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For example, f_d are sharp. Question is, what are all the sharp examples. It is not too difficult to show that there are only finitely many sharp polynomials in $\mathcal{H}(2, d)$. We will say that *uniqueness holds* in degree d, if there is only one sharp polynomial up to swapping of variables.

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Theorem (– and D'Angelo, '09)

There are infinitely many odd degrees for which uniqueness does not hold. Also, uniqueness does not hold for all even degrees.

What we did, I

What we did (– and Lichtblau) is to classify all sharp polynomials up to $d = 17$. I will WOW you with a table on the next slide. In terms of uniqueness we proved

Theorem

For $d \leqslant 17$, uniqueness holds for $d = 1, 3, 5, 9, 17$.

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Theorem

For $d \leqslant 17$, uniqueness holds for $d = 1, 3, 5, 9, 17$.

There is a procedure for generating new examples using f_d so we have a list of candidates for uniqueness (where the procedure does not apply). This list is

$$1, 3, 5, 9, 17, 21, 33, 41, 45, 53, 69, 77, 81, 93, 105, 113, 117, 125, 129, 141, 149, 153, 161, 165, 177, 185, 201, 213, 221, 225, 249, 261, 269, 273, 285, 297, 305, 309, 333, 341, 345, 357, 365, 369, 381, 405, 413, 417, 429, 437, 441, 453, 465, 473, 489, 501, \ldots$$

What we did, II

The following table lists all sharp polynomials in odd degrees up to swapping of variables. A $*$ next to degree indicates uniqueness.

d	Sharp polynomials
1^*	$x + y$
3^*	$x^3 + 3xy + y^3$
5^*	$x^5 + 5x^3y + 5xy^2 + y^5$
7	$x^7 + 7x^3y + 14x^2y^3 + 7xy^5 + y^7$
	$x^7 + 7x^3y + 7x^3y^3 + 7xy^3 + y^7$
	$x^7 + \frac{7}{2}x^5y + \frac{7}{2}xy + \frac{7}{2}xy^5 + y^7$
9^*	$x^9 + 9x^7y + 27x^5y^2 + 30x^3y^3 + 9xy^4 + y^9$
11	$x^{11} + 11x^9y + 44x^7y^2 + 77x^5y^3 + 55x^3y^4 + 11xy^5 + y^{11}$
	$x^{11} + 11x^5y + 11x^5y^5 + 55x^4y^3 + 55x^3y^5 + 11xy^5 + y^{11}$

Continued ...

What we did, III

d	Sharp polynomials
13	$x^{13}+13x^{11}y+65x^9y^2+156x^7y^3+182x^5y^4+91x^3y^5+13xy^6+y^{13}$
	$x^{13}+13x^{11}y+65x^9y^2+\frac{221}{2}x^7y^3+\frac{92}{2}x^3y^3+\frac{91}{2}x^3y^7+13xy^6+y^{13}$
	$x^{13}+\frac{234}{25}x^{11}y+\frac{143}{5}x^8y^2+\frac{143}{5}x^7y^4+\frac{91}{25}xy+\frac{143}{25}xy^6+\frac{91}{25}xy^{11}+y^{13}$
	$x^{13}+\frac{234}{25}x^{11}y+\frac{143}{5}x^9y^2+\frac{143}{5}x^7y^3+\frac{91}{25}xy+\frac{143}{25}xy^6+\frac{91}{25}xy^{11}+y^{13}$
15	$x^{15}+15x^{13}y+90x^{11}y^2+275x^9y^3+450x^7y^4+378x^5y^5+140x^3y^6$ $+15xy^7+y^{15}$
	$x^{15}+140x^9y^3+15x^7y+420x^7y^4+15x^7y^7+378x^5y^5+140x^3y^6$ $+15xy^7+y^{15}$
17*	$x^{17}+17x^{15}y+119x^{13}y^2+442x^{11}y^3+935x^9y^4+1122x^7y^5$ $+714x^5y^6+204x^3y^7+17xy^8+y^{17}$

$d=19$ will take 1 year on my machine, but NSF will buy me a new computer come fall, which should drop it into the realm of a month or two. For $d=19$, there are at least two other sharp polynomials besides f_d that we know.

Even degrees

In even degrees, we showed that up to $d = 12$, sharp polynomials are constructed by the following procedure. Take $p \in \mathcal{H}(2, a)$ and $q \in \mathcal{H}(2, b)$ with $a + b = d$. Then construct a polynomial

$$p(x, y) - x^a + x^a q(x, y).$$

It seems reasonable to conjecture that all even degree examples come from this procedure.

Note that the number of even degree examples goes to infinity as d goes to infinity.

How we did it

Notice that $p(x, 1-x) - 1$ equals zero for all x. Hence we get a set of linear equations of the coefficients. As the coefficients also have to be nonnegative, this is a natural linear programming problem.

We wrote two pieces of code. One is a mixed-integer programing approach. One is a linear algebra approach, which at first ignores the inequalities.

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First find an upper bound m_{jk} for every coefficient c_{jk} of the monomial $x^j y^k$. Then take

$$0 \leqslant c_{jk} \leqslant m_{jk} b_{jk}$$

where m_{jk} is the upper bound and $b_{jk} \in \{0, 1\}$ is a variable. We use the linear equations to eliminate variables and adjust the inequalities.

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We work with a stack of problems. We push a relaxed version of the problem where $b_{jk} \in [0, 1]$. We solve. If a variable b_{jk} is not 0 or 1 we create and push two new problems with this requirement. Once we get a valid solution with $b_{jk} \in \{0, 1\}$ for all j and k, then we push new problems by in turn forcing one of the b_{jk} to 0.

Linear algebra approach

We could also just look at the equations coming from $p(x, 1-x) - 1 \equiv 0$ and also force a certain set of the coefficients $c_{jk} = 0$. Note that the -1 gives us an inhomogeneous problem. So we can easily solve by Gaussian elimination. However, we get too many solutions.

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This approach was prototyped in the *Genius* software (my own free software package). Then the algorithm was rewritten in C using the *GMP* library for large integer support.

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Some further minor simplifications are also done, but the above are the main ones.

Another optimization

For the linear algebra approach, before computing the nullspace with integer arithmetic, what we do is to first compute it in a small finite field. Surprisingly, $\mathbb{F}_{19}$ was sufficient. That way we can easily throw out cases where the matrix was nonsingular.

Timings

The $d = 17$ case was on the order of days for both methods.

The mixed-integer programing method scales better. Around 50 fold for increasing degree by 2. The linear algebra approach seems to increase 100 fold.

Some URLs

Code for the algorithms: `http://www.jirka.org/LL08-archive.zip`

Genius: `http://www.jirka.org/genius.html`

Mathematica: `http://www.wolfram.com/`

Thank you!